Temporal intermittency and cascades in shell models of turbulence

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The two-dimensional- (2D) and three-dimensional- (3D) like Gletzer, Okhitani, and Yamada shell models are examined. The 2D-like model shows a transition from statistical quasiequilibrium to cascade of enstrophy as a function of the spectral ratio of energy to enstrophy. The transition is related to the ratio of time scales, corresponding to eddy turnover times, between shells. The anomalous scaling, giving rise to nonlinear scaling functions, is also connected to the ratio of eddy turnover times. This is illustrated in a simple stochastic model, where the structure function $\zeta(q)$ becomes independent of q. In the 3D-like model the multiscaling is also influenced by the existence of a second nonpositive-definite inviscid invariant, the helicity. [S1063-651X(96)09107-6]

The problem of understanding the scaling properties of velocity correlations in isotropic and homogeneous turbulence is still largely unsolved. The only exact derived property is the Kolmogorov [1] scaling law for the third-order moments $C_3(r) = -4\epsilon r/5 + 6\nu dC_2(r)/dr$, where $C_q(r) \equiv \langle \Delta v(r)^q \rangle = \langle |v(x+r) - v(r)|^q \rangle$, r is the distance in the fluid, and $\langle \rangle$ is the ensemble average. This relation is not closed since it contains both second- and third-order correlation functions. In the inertial range defined by $L \gg r \gg \lambda$, where L is the outer scale and λ the inner or Kolmogorov dissipation scale, the second term on the right-hand side is negligible and the scaling relation, $C_3(r) \sim r^{\zeta(3)}$, with $\zeta(3) = 1$, holds. The classical Kolmogorov theory [1(a)] based on dimensional arguments states that $C_q(r) \sim r^{\zeta(q)}$, with $\zeta(q) = q/3$. It was noted by Landau [2] shortly after the theory was presented that the energy dissipation could vary so much as to alter the scaling laws. This was incorporated in a refined version of Kolmogorovs theory [1(b)], in which a dependence on L/r, the ratio of the outer scale to the scale in the inertial range, was incorporated.

It has been seen in numerous experiments that $\zeta(q)$ is a weakly nonlinear function of q, different from the Kolmogorov predictions for q different from 3. It follows from the Hølder inequality that $\zeta(q)$ is a convex of q, thus $\zeta(q) > q/3$ for $q < 3$ and $\zeta(q) < q/3$ for $q > 3$. The deviation of the scaling from the [1(a)] prediction is referred to as intermittency corrections. It is widely attributed to the fact that the energy dissipation in fully developed turbulence is indeed highly inhomogeneous, basically taking place on lower dimensional subsets, filaments, of the flow field.

Numerical studies of the Gletzer-Okhitani-Yamada (GOY) shell model [3] of turbulence have been popular resently, mainly because this model, in which scaling relations are completely equivalent to those of turbulence, also shows intermittency corrections to the [1(a)], equivalent, predictions. The GOY model structurally resembles the spectral

form of the Navier-Stokes equation, but there are no spatial fields associated with the wave components in the GOY model. The intermittency corrections in the scaling properties are thus associated with temporal intermittency, which can only be weakly linked to spatial intermittency through the Taylor hypothesis. The connection between the spatial intermittency of turbulence and temporal intermittency of the GOY model is not clear, but the hope is that understanding the latter can shed light upon the former. This note is about the latter.

In the GOY model the spectral domain is represented as shells, each of which is defined by a wave number $k_n = \lambda^n$, where λ is a scaling parameter defining the shell spacing. There are $2N$ degrees of freedom, where N is the number of shells, namely, the generalized complex shell velocities, u_n for $n = 1, N$. The dynamical equation for the shell velocities is

$$\dot{u}_n = ik_n \left(u_{n+2} u_{n+1} - \frac{\epsilon}{\lambda} u_{n+1} u_{n-1} + \frac{(\epsilon - 1)}{\lambda^2} u_{n-1} u_{n-2} \right)^* - \nu k_n^2 u_n + f \delta_{n,n_0}, \tag{1}$$

where the first term represents the nonlinear wave interaction or advection, the second term is the dissipation, and the third term is the forcing, where n_0 is some small wave number. The boundary conditions are $u_{-1} = u_0 = u_{N+1} = u_{N+2} = 0$. The model has two quadratic inviscid invariants, which are constants of motion in the case $f = \nu = 0$, $E = \Sigma_n |u_n|^2$, referred to as the energy, and $H = \Sigma_n (\epsilon - 1)^{-n} |u_n|^2$. For $\epsilon < 1$, H is nonpositive definite, referred to as the helicity, and the model is thought of as modeling 3D turbulence. For $\epsilon > 1$, H is positive definite, referred to as the enstrophy, and the model is meant to resemble 2D turbulence. Dimensional arguments similar to those of [1(a)] can be applied to the GOY model assuming dissipation of one of the conserved quantities. For the 2D-type model, $1 < \epsilon < 2$, enstrophy will be dissipated and an additional large-scale drag term $-\nu' k_n^{-2} u_n$ is added to (1) in order to remove the energy. The reason for the enstrophy and not the energy being dissipated is, as usual, that the dissipation of energy is $k_n^{-\alpha}$ times the dissipation of enstrophy at shell n, thus negligible for $\alpha > 0$ and $n \to \infty$. The Kolmogorov scaling for the 2D-type model then becomes $\langle |u_n| \rangle \sim k_n^{-(\alpha+1)/3}$. Note that this is an

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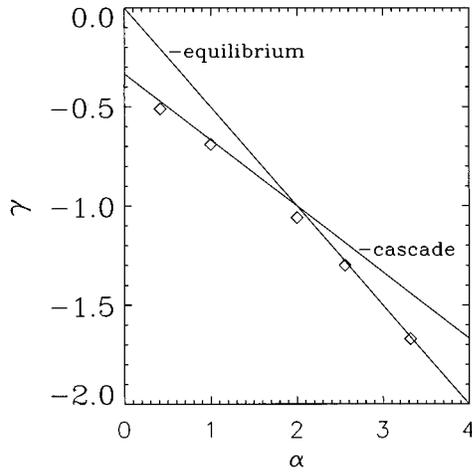
FIG. 1. Results of runs of the 2D-like GOY model for different values of ϵ. Shown is the spectral slope $\gamma = -\log(\langle u \rangle)/\log(k)$ as function of $\alpha = -\log(\epsilon-1)/\log(\lambda)$. The lines are the Kolmogorov scaling for enstrophy cascade and equipartitioning of enstrophy, respectively. For the 2D-like GOY models an additional dissipation of the form $-\nu' k_n^{-2} u_n$ is applied in order to remove energy at small wave-number shells. The model is run with $n=50$, $k_0=\lambda^{-4}$, $\lambda = 2$, $f_n = 5 \times 10^{-4} \times (1+i)\delta_{n,15}$, ($\epsilon = 11/10$, $\nu = 5 \times 10^{-27}$, $\nu' = 100$), ($\epsilon = 117/100$, $\nu = 5 \times 10^{-27}$, $\nu' = 100$), ($\epsilon = 5/4$, $\nu = 5 \times 10^{-25}$, $\nu' = 100$), ($\epsilon = 3/2$, $\nu = 5 \times 10^{-23}$, $\nu' = 100$), ($\epsilon = 7/4$, $\nu = 5 \times 10^{-23}$, $\nu' = 100$), and ($\epsilon = 2$, $\nu = 5 \times 10^{-20}$, $\nu' = 100$). The models show a crossover at $\alpha = 2$ between statistical equilibrium $\alpha > 2$ and enstrophy cascading $\alpha < 2$.

unstable fixed point of (1) for $f = \nu = 0$, describing a cascade. As pointed out by Aurell *et al.* [4], in the case of $\alpha = 2$ this scaling coincides with the scaling that would be obtained in a statistical equilibrium, in which the enstrophy is distributed evenly among the degrees of freedom of the system $\langle |u_n| \rangle \sim k_n^{-\alpha/2}$. The $\alpha = 2$, corresponding to $\epsilon = 5/4$, case is a borderline case between models showing cascade, for $\alpha < 2$, and models showing statistical (quasi)equilibrium, for $\alpha > 2$. In order to show this a series of numerical model runs for different values of ϵ have been performed, details of which are reported elsewhere [5]. Figure 1 shows the spectral slopes obtained. The two lines shows the slopes corresponding to cascade and statistical equilibrium, respectively, and the diamonds are the numerical results. The figure clearly shows a crossover from one type of behavior to the other. The reason for this transition is related to the scaling of typical time scales, or eddy turnover times, for the different shells. The eddy turnover time at shell n is given as $\tau_n = (k_n \langle |u_n| \rangle)^{-1}$, as seen from (1) or dimensional arguments. The eddy turnover time then becomes $\tau_{\text{cascade}} \sim k^{(\alpha-2)/3}$ and $\tau_{\text{equilibrium}} \sim k^{(\alpha-2)/2}$, respectively. Thus, for $\alpha > 2$ the eddy turnover times in both cases increase with wave number and equilibration via inverse enstrophy transfer, from large-wave-numbered shells to small-wave-numbered, shells takes place. For $\alpha < 2$ the situation is reversed and a statistical equilibrium can never be achieved. Similar results have been found by Yamada and Ohkitani [6] for a slightly different set of GOY models, with only one positive-definite inviscid invariant, referred here to as 3D-type models.

In the 3D-type models the situation is totally different. In

this case the second inviscid invariant, the helicity, is not cascaded even though the ratio of the absolute value of the helicity to the energy at shell n grows exponentially with n, as is the case for the enstrophy. The reason is that the helicity has alternating signs for even and odd numbered shells. Therefore there will be a net production of (positive sign) helicity at odd numbered shells due to the dissipation and no net flow of helicity from small-wave-number shells to large-wave-number shells is necessary. It has been demonstrated numerically that helicity is indeed the quantity to be cascaded in the (pathological) case of hyperviscosity only active on the outermost shell [7]. In the usual 3D case energy will be cascaded, with the resulting spectral scaling $\langle |u|^3 \rangle \sim k^{-1}$.

Both the 2D and the 3D models shows intermittency corrections to the Kolmogorov scaling depending on ϵ and λ. In this paper I will suggest two different mechanisms in play in the 3D case, where only the one is in action in the 2D case.

In the 3D case it is observed that the structure function depends only on ϵ and λ in the combination, $-\log(1-\epsilon)/\log(\lambda) = \alpha$ [8]. With $\delta\zeta(q) \equiv \zeta(q) - q/3$, $\delta\zeta(q)$ increases in absolute value when α increases. I suggest that this is due to differences in the ratio of helicity production and helicity elimination by dissipation at neighboring shells in the beginning of the dissipative subrange where scaling still approximately holds or where extended self-similarity
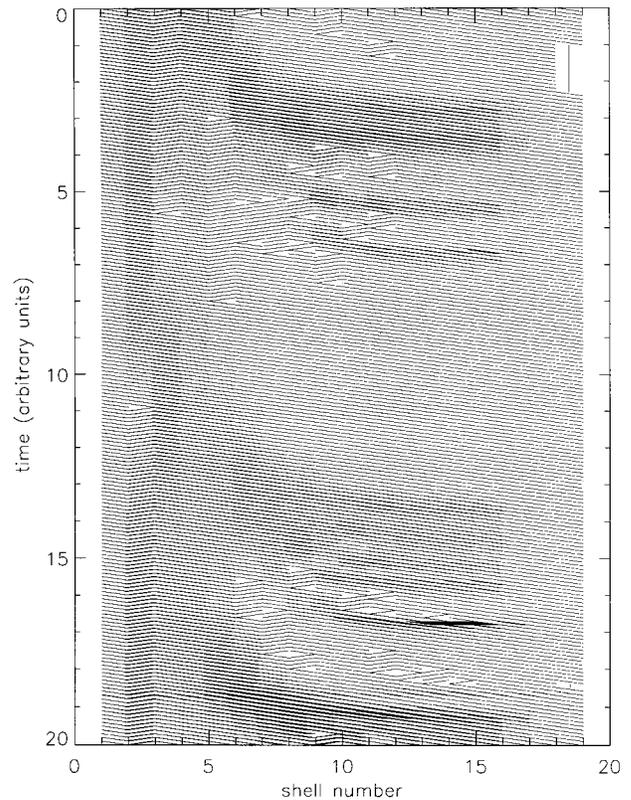
FIG. 2. Nonlinear transfer of energy from one shell to the next in the 3D-like GOY model with $\epsilon = 1/2$. All parameter values are standard and forcing is on shell 4. The line segments pointing from shell n at time t to shell $n+1$ at time $t + \Delta t$ symbolize forward transfer, with magnitude proportional to the thickness of the line segment. Line segments pointing from shell $n+1$ at time t to shell n at time $t + \Delta t$ symbolize backward transfer.
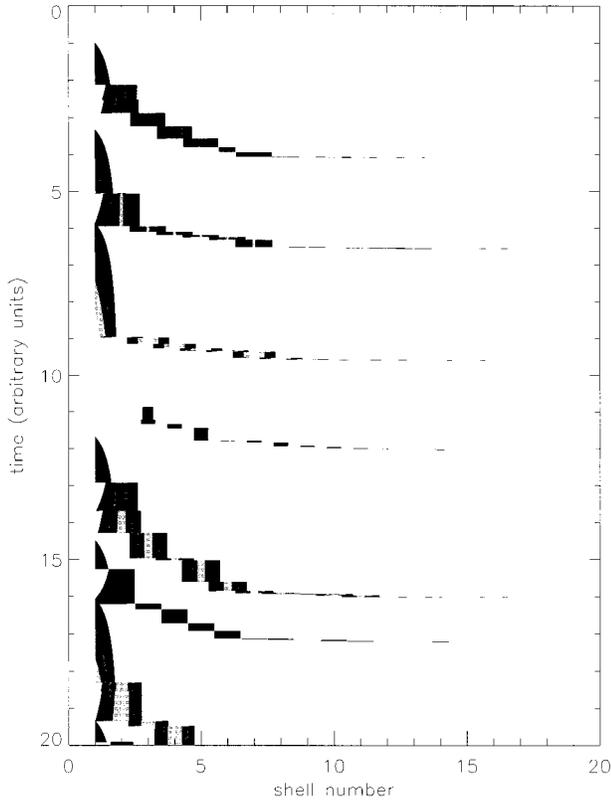
FIG. 3. Result of a simulation of the simple stochastic model. x_n is plotted as a function of time. The intermittent transfer is similar to what is seen in the GOY model, Fig. 2.

can be applied [9]. This ratio r can be estimated as $r \equiv (\Delta H_{n-1} + \Delta H_{n+1})/2\Delta H_n \sim (\lambda^{-\alpha+2/3} + \lambda^{\alpha-2/3})/2$, which for $(\alpha, \lambda) = (0.5,2),(1,2),(2,2)$ gives $1.007,1.03,1.46$, respectively. So there is the largest ''mismatch,'' or noncancellation, in the case $(2,2)$ where the largest nonlinearity in the structure function is observed. That these two things should be related is consistent with the findings of a moderated GOY model (model 3) by Benzi *et al.* [10], where two copies of the GOY model are coupled. In this model the helicity takes the form $H = \Sigma_n k_n^\alpha (|u_n^+|^2 - |u_n^-|^2)$, where u_n^+ and u_n^- are the two complex variables in shell n. In this model the helicity production and elimination will, on average, exactly balance, thus no dependence of the structure function on α should be expected in agreement with the numerical findings.

In the standard 3D case $\alpha = 1$, the structure function is still nonlinear even though the above defined r is close to 1. This is suggested to be due to the difference in time scales between the large and the small scales. This effect is expected also to determine the anomalous scaling in the 2D case, where only the dissipation of enstrophy plays a role. In the 2D case numerical studies suggests that the absolute size of $\delta\zeta(q)$ increases as α decreases from 2 to 0 [5]. This suggests that intermittency is enhanced with growing difference between eddy turnover times from one shell to the next, which goes as $\tau_{n+1}/\tau_n = \lambda^{(2-\alpha)/3}$. Figure 2 shows the nonlinear transfer $\Delta_n = k_{n-1} \mathrm{Im}(u_{n-1} u_n u_{n+1})$ of energy from shell $n-1$ to shells n and $n+1$ in the 3D case. The abscissa is the shell number and the ordinate is time, read from top to

bottom. The figure is composed of line segments connecting shells n at time t_i and shells $n+1$ at time $t_i + \Delta t$, symbolizing an energy transfer from shell n in the time interval $[t_i ; t_i + \Delta t]$. The thickness of the line segment is proportional to the size of the transfer. Line segments going from $n+1$ at time t_i to n at time $t_i + \Delta t$ symbolize the inverse energy transfer. The figure shows that the transfer is temporally intermittent and occurring in bursts [11], with the transfer being faster as the bursts propagate to higher-wavenumber shells. From this the interpretation is straightforward; the residence time for a burst at a given shell is proportional to the eddy turnover time resulting in a more and more intermittent transfer as the eddy turnover time decreases. In order to illustrate how this kind of behavior can lead to anomalous scaling behavior, consider the following linear stochastic model, which is an extreme case.

Let x_n be a stochastic variable representing the energy at shell n, governed by the dynamical equation

$$x_n(t+1) = p_{n-1}(t)x_{n-1}(t) - p_n(t)x_n(t) - \nu k_n^2 x_n(t) + f\delta_{n,1}, \tag{2}$$

where p_n is a stochastic variable that is 1 with probability τ_N/τ_n and 0 with probability $1 - \tau_N/\tau_n$; thus it represents a transition probability of energy transfer from shell n to shell $n+1$. The boundary conditions are $x_{-1} = p_N = 0$. The average residence time is then simply proportional to the eddy turnover time. It is easily seen that x_n is always positive.
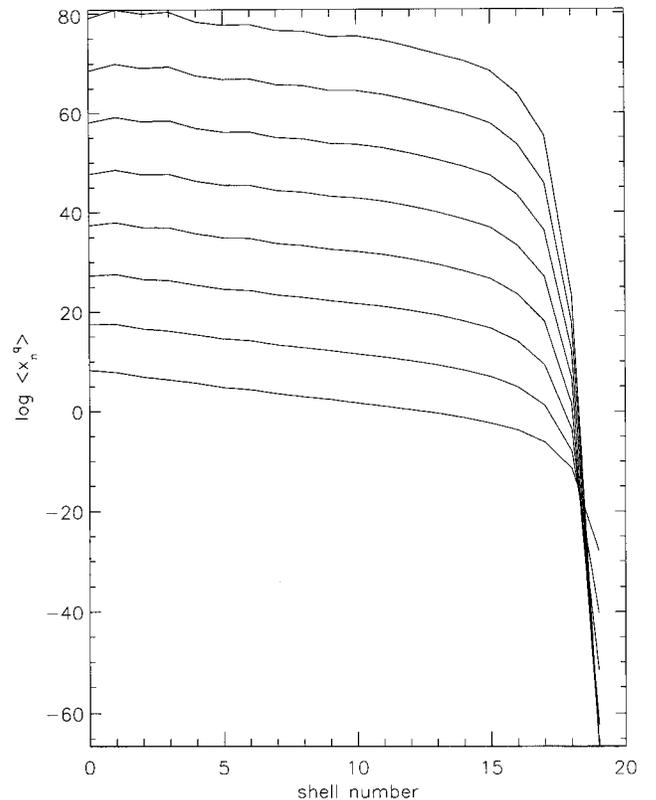
FIG. 4. Anomalous scaling behavior of the simple stochastic model. $\log_2(\langle x^q \rangle)$ vs $\log_2(k)$ for $q = 1,8$. The spectral slope does not depend on q so that $\zeta(q) = (\alpha - 2)/3$. The model was iterated for 10^4 large eddy turnover times with $N = 20, f = 0.05$, and $\nu = 10^{-11}$.

Figure 3 is similar to Fig. 2, but shows the values of x_n. There is obviously no inverse transfer in this model. By comparing Figs. 2 and 3 one sees that the intermittent structures of the transfers are similar. The energy will, in this model, scale inversely proportionally to the eddy turnover time, so that if we identify x_n with the energy or enstrophy of the GOY model $E_n = k_n^\alpha |u_n|^2$, the scaling of $\langle |u_n| \rangle \sim k_n^{-(\alpha+1)/3}$ is the same as for the GOY model. However, the structure function changes completely. It is readily calculated, with the result $\langle x_n^q \rangle \sim k_n^{(\alpha-2)/3} x^q$, where $\tau_n = k_n^{(\alpha-2)/3}$ is the eddy turnover time at shell n and $x = f\tau_1$ is the average energy input into the system during one large eddy turnover time. This means that the structure function $\zeta(q)$ becomes independent of q. q will only show up in the offset, $q\log(x)$ in a $\log\langle x_n^q \rangle$, $\log(k_n)$ plot. This is illustrated in Fig. 4, showing the result of a numerical simulation. Obviously this linear stochastic model cannot reproduce the scaling exponents of the GOY model or of real turbulence, but it qualitatively illustrates the effect of temporal intermittency.

In conclusion, the behavior of the 2D-like GOY model shows either statistical equilibrium or cascade of enstrophy depending on the ratio of the eddy turnover time scales between the shells. The dynamics and the multiscaling of the 3D-like GOY model depends on the existence of the second nonpositive-definite inviscid invariant, helicity, through the way the helicity is dissipated in the model. In both cases, the multiscaling is an effect of temporal intermittency also originating from differences in eddy turnover times at the different scales; thus the 2D model with $\alpha = 2$ shows no anomalous scaling. The effect is illustrated in a simple stochastic model. The scaling behavior of this simple model does not correspond to what is seen in the GOY model or in real turbulence. However, the model points to a mechanism of how the temporal intermittency can lead to anomalous scaling behavior. In order to further clarify the relationship between the spatial intermittency observed in turbulence and the temporal intermittency seen in these simple models, it would be interesting to see if the structures seen in the time–wave-number domain, Fig. 1, can be identified in direct numerical simulations of the Navier-Stokes equation.

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